BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 35(2), 177-198

On the horospherical ridges of submanifolds of codimension 2 in Hyperbolic n-space

S. Izumiya, D. Pei, M.C. Romero-Fuster*
and M. Takahashi

Abstract. We introduce the notion of horospherical ridges for submanifolds of codimension 2 in hyperbolic n-space, and study some of their properties.

Keywords: horospherical ridge point, hyperhorosphere, horoasymptotic line, hyperbolic height function.

Mathematical subject classification: 52S05, 53A35.

1 Introduction

The ridges of surfaces in 3-space were introduced by I. Porteous [7] as the sets of points at which the surface has a higher order contact with some of their focal spheres. They are the image, through the exponential map, of the singular part of the focal set off the umbilical foci. The generic structure of these sets may be described by means of the analysis of the singularities of the distance squared functions on the surface. The generalization of these ideas to hypersurfaces in $\mathbb{R}^n$ follows in a natural way (cf., [9]). The analogous study, applied to the singularities of the height functions over submanifolds of codimension 2 in $\mathbb{R}^n$, leads to the concept of flat ridges. They were introduced by the third named author and E. Sanabria-Codesal in [8] and are made of points at which the submanifolds have higher order contact with some hyperplane. In fact, the flat ridges of an $(n-2)$-submanifold M coincide with the image of the intersection locus of the ridges and the parabolic subset of the canal hypersurface CM through the natural projection $\nu\colon CM \to M$. The flat ridges may be classified in terms of the order of contact of the submanifold with the hyperplanes. Those of maximal order are

Received 12 March 2003.
*Work partially supported by DGCYT grant no. BFM2003-02037.

isolated points over generic submanifolds. An interesting fact, shown in [8], is that these can be characterized as flattenings of the asymptotic lines, considered as curves in the ambient Euclidean space. Asymptotic lines of $(n-2)$-submanifolds of $\mathbb{R}^n$ were introduced in [6].

The purpose of this paper is to develop the analogous theory in the context of the submanifolds of hyperbolic spaces and their contacts with hyperhorospheres. To do this, we introduce in Section 3 the *hyperbolic height functions family* on $(n-2)$-submanifolds of $H^n_+(-1)$. This functions measure the contacts of such submanifolds with hyperhorospheres in $H^n_+(-1)$ in a similar way than the height functions do for submanifolds and hyperplanes in $\mathbb{R}^n$. We define the *hyperbolic canal hypersurface, CM,* of M in $H^n_+(-1)$ and relate the singularities of hyperbolic height functions on M with those on CM. In fact, we see that the degenerate singularities of the hyperbolic height functions on M correspond to the singularities of the hyperbolic Gauss map on CM under the natural projection $\nu : CM \to M$ (Proposition 3.3). This setting allows us to define the concepts of *horobinormals* and *osculating hyperhorospheres* at the points of M, as well as, *horospherical ridges*. The horospherical ridge points are the singularities of type $A_{k\geq 3}$ of hyperbolic height functions on M. They correspond to degenerate singularities of corank one for the hyperbolic Gauss map on CM under ν. We see that they are contained in the closure, $cl(M^*)$, of an open submanifold, M^*, of M and form regular submanifolds of codimension one when restricted to M^* (Proposition 3.9). In Section 4 we define the horoasymptotic directions on submanifolds of codimension 2 of hyperbolic n-space that generalize those defined in [4] for surfaces in $H^4_+(-1)$. We define *horospherical flattenings* of a curve in hyperbolic n-space as those at which the curve has a contact of order $n+1$ with some hyperhorosphere. Then the ridge points of M are characterized as horospherical flattenings of the normal sections of M in horoasymptotic directions (Theorem 4.2). The horoasymptotic directions determine some tangent fields over the closure of M^* whose integral curves are the *horoasymptotic lines* of M.

The main result in this section consists in the characterization of the maximal order horospherical ridges of a submanifold M of codimension 2 in hyperbolic n-space as the horospherical flattenings of its horoasymptotic lines, considered as curves in $H^n_+(-1)$ (Corollary 4.8).

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Basic notions and concepts

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \dots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, \dots, x_n),\ \boldsymbol{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski* $(n+1)$*–space* and written by $\mathbb{R}^{n+1}_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike, lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{n}$ is given by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike, timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

The *hyperbolic* n*-space* is defined by

$$H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 \geq 0\}$$

and the *de Sitter* n*-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}.$$

Given n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \in \mathbb{R}^{n+1}_1$, we can define another vector $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ as follows:

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ a^1_0 & a^1_1 & \cdots & a^1_n \\ a^2_0 & a^2_1 & \cdots & a^2_n \\ \vdots & \vdots & \cdots & \vdots \\ a^n_0 & a^n_1 & \cdots & a^n_n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{a}_i = (a^i_0, a^i_1, \dots, a^i_n)$. We can easily check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n),$$

so $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo orthogonal to $\boldsymbol{a}_i$ for $i = 1, \dots, n$.

We also define a set $LC_a = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\ \}$, which is called *a closed lightcone* with the vertex $\boldsymbol{a}$. We denote

$$LC^*_+ = \{\boldsymbol{x} = (x_0, \dots x_n) \in LC_0\ | x_0 > 0\ \}$$

and we call it *the future lightcone* at the origin. Given any lightlike vector $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, we have that $x_0 \neq 0$ and thus,

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in LC^*_+ \mid x_0 = 1\}.$$

The subset S^{n-1}_+ is known as *the spacelike* $(n-1)$*-sphere*.

Let $U \subset \mathbb{R}^{n-2}$ be an open subset and suppose that $\boldsymbol{x} : U \to H^n_+(-1)$ is an embedding, so its image $M = \boldsymbol{x}(U)$ is a regular submanifold of codimension 2 in $H^n_+(-1)$. We shall identify M with U by the embedding $\boldsymbol{x}$.

Given $p = \boldsymbol{x}(u) \in M \subset H^n_+(-1)$, we have $\langle \boldsymbol{x}(u), \boldsymbol{x}(u)\rangle = -1$, so that $\langle \boldsymbol{x}_i(u), \boldsymbol{x}(u)\rangle = 0$, where $u = (u_1, u_2, \dots, u_{n-2})$ and $\boldsymbol{x}_i(u) = (\partial \boldsymbol{x}/\partial u_i)(u)$. Hence the tangent space of M at $p = \boldsymbol{x}(u)$ is

$$T_pM = \langle \boldsymbol{x}_1(u), \boldsymbol{x}_2(u), \dots, \boldsymbol{x}_{n-2}(u)\rangle_{\mathbb{R}}.$$

Let N_pM be the pseudo normal space of M at $p = \boldsymbol{x}(u)$ in $\mathbb{R}^{n+1}_1$ and choose a pseudo normal vector $\boldsymbol{n}(u) \in S^1(N_pM \cap T_pH^n_+(-1))$, where $S^1(N_pM \cap T_pH^n_+(-1))$ denotes the spacelike unit circle in $N_pM \cap T_pH^n_+(-1)$. We remark that $u \in U$ is not necessarily fixed, so that $\boldsymbol{n}(u)$ might be a pseudo normal vector field along M. It follows that

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \cdots \wedge \boldsymbol{x}_{n-2}(u) \wedge \boldsymbol{n}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \cdots \wedge \boldsymbol{x}_{n-2}(u) \wedge \boldsymbol{n}(u)\|} \in S^1(N_pM \cap T_pH^n_+(-1)).$$

Then we have the following Proposition:

Proposition 2.1. *Under the above notations, we have*

$$N_pM = \langle \boldsymbol{x}(u), \boldsymbol{n}(u), \boldsymbol{e}(u)\rangle_{\mathbb{R}}.$$

Moreover,

$$\boldsymbol{n}_i(u) \in \langle \boldsymbol{x}_1(u), \boldsymbol{x}_2(u), \dots, \boldsymbol{x}_{n-2}(u), \boldsymbol{e}(u)\rangle_{\mathbb{R}}$$

and

$$\boldsymbol{e}_i(u) \in \langle \boldsymbol{x}_1(u), \boldsymbol{x}_2(u), \dots, \boldsymbol{x}_{n-2}(u), \boldsymbol{n}(u)\rangle_{\mathbb{R}},$$

where $\boldsymbol{n}_i(u) = (\partial \boldsymbol{n}/\partial u_i)(u)$, $\boldsymbol{e}_i(u) = (\partial \boldsymbol{e}/\partial u_i)(u)$.

Proof. By the above construction, $\{\boldsymbol{x}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-2}, \boldsymbol{n}, \boldsymbol{e}\}$ is a basis of the vector space $T_p\mathbb{R}^{n+1}_1$, and thus we can write $\boldsymbol{n}_i = \sum_{j=1}^{n-2} \lambda^i_j \boldsymbol{x}_j + \mu^i_1 \boldsymbol{n} + \mu^i_2 \boldsymbol{e} + \mu^i_3 \boldsymbol{x}$, for some $\lambda^i_j, \mu^i_k \in \mathbb{R}$, $i, j = 1, 2, \ldots, n-2$ and $k = 1, 2, 3$. Since $\langle \boldsymbol{n}, \boldsymbol{n}\rangle = 1$, $\langle \boldsymbol{n}_i, \boldsymbol{n}\rangle = 0, i = 1, \ldots, n-2$. Thus we have $\mu^i_1 = 0, i = 1, \ldots, n-2$. On the other hand, $\langle \boldsymbol{x}, \boldsymbol{n}\rangle = 0$ and hence $\langle \boldsymbol{x}, \boldsymbol{n}_i\rangle = -\langle \boldsymbol{x}_i, \boldsymbol{n}\rangle = 0, i = 1, \ldots, n-2$. Therefore $\mu^i_3 = 0, i = 1, \ldots, n-2$. Consequently we get

$$\boldsymbol{n}_i(u) \in \langle \boldsymbol{x}_1(u), \boldsymbol{x}_2(u), \ldots, \boldsymbol{x}_{n-2}(u), \boldsymbol{e}(u)\rangle_{\mathbb{R}}.$$

The final assertion follows similarly. □

We shall consider a fixed unit pseudo normal vector field $\boldsymbol{n}$ on $M \subset H^n_+(-1)$ by the above construction through the whole paper.

3 Hyperbolic height functions on submanifolds of codimension 2

In this section we introduce the notions of hyperbolic height functions, hyperbolic canal hypersurfaces and hyperbolic Gauss maps on submanifolds of codimension 2 in $H^n_+(-1)$.

For a regular submanifold $M(= \boldsymbol{x}(U))$ of codimension 2 in $H^n_+(-1)$, we define a function

$$H\colon U \times S^{n-1}_+ \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle$, where $\boldsymbol{v} = (1, v_1, \ldots v_n) \in S^{n-1}_+$. We call H the *hyperbolic height functions family* on M. We shall denote $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$, for any $\boldsymbol{v}_0 \in S^{n-1}_+$. Then we have the following proposition:

Proposition 3.1. *Let M be a regular submanifold of codimension 2 in $H^n_+(-1)$ and $H\colon U \times S^{n-1}_+ \longrightarrow \mathbb{R}$ the hyperbolic height function. Then $(\partial h_v/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-2)$ if and only if $\boldsymbol{v} = \widetilde{\boldsymbol{x} + \boldsymbol{w}}(u, \varphi)$, where $\boldsymbol{w}(u, \varphi) = \cos\varphi \boldsymbol{n}(u) + \sin\varphi \boldsymbol{e}(u)$ and $0 \le \varphi < 2\pi$.*

Proof. We remark that $(\partial h_v/\partial u_i)(u) = 0$ if and only if $\langle \boldsymbol{x}_i, \boldsymbol{v}\rangle = 0$ for each $i = 1, \ldots, n-2$ and $\boldsymbol{v} \in S^{n-1}_+$. This is equivalent to the condition that $\boldsymbol{v} \in \langle \boldsymbol{x}, \boldsymbol{n}, \boldsymbol{e}\rangle_{\mathbb{R}} \cap S^{n-1}_+$. Therefore, $\boldsymbol{v} = \lambda\boldsymbol{x} + \mu\boldsymbol{n} + \xi\boldsymbol{e}$ for some real numbers λ, μ, ξ. Since $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, $\mu^2 + \xi^2 = \lambda^2$. It follows that

$$\lambda\left(\boldsymbol{x} + \frac{\mu}{\lambda}\boldsymbol{n} + \frac{\xi}{\lambda}\boldsymbol{e}\right) = \lambda(\boldsymbol{x} + \cos\varphi\boldsymbol{n} + \sin\varphi\boldsymbol{e}),$$

where $\mu/\lambda = \cos\varphi$ and $\xi/\lambda = \sin\varphi$. This means that $\boldsymbol{v} = \widetilde{\boldsymbol{x} + \boldsymbol{w}}(u, \varphi)$, where $\boldsymbol{w}(u, \varphi) = \cos\varphi \boldsymbol{n}(u) + \sin\varphi \boldsymbol{e}(u)$ and $0 \leq \varphi < 2\pi$. The converse follows from straightforward calculations. □

The *hyperbolic canal hypersurface* of M in $H^n_+(-1)$ is defined by

$$\begin{aligned} CM \ = \ \Big\{ & (\boldsymbol{x}(u), \boldsymbol{v}) \in M \times S^{n-1}_+ \, | \boldsymbol{v} = \widetilde{\boldsymbol{x} + \boldsymbol{w}}(u, \varphi), \\ & \boldsymbol{w} \in S^1(N_{\boldsymbol{x}(u)} M \cap T_{\boldsymbol{x}(u)} H^n_+(-1)) \Big\}. \end{aligned}$$

Let $DH \colon M \times S^{n-1}_+ \longrightarrow \mathbb{R} \times S^{n-1}_+$; $DH(\boldsymbol{x}(u), \boldsymbol{v}) = (H(u, \boldsymbol{v}), \boldsymbol{v})$ be the unfolding associated to the family H. The singular set of DH is given by

$$\sum(DH) = \Big\{ (\boldsymbol{x}(u), \boldsymbol{v}) \in M \times S^{n-1}_+ | \ \langle d\boldsymbol{x}, \boldsymbol{v} \rangle = 0 \Big\}.$$

It follows from Proposition 3.1 that $\sum(DH) = CM$.

We can consider the hyperbolic canal hypersurface of M, as a hypersurface of $H^n_+(-1)$ by means of an embedding, $\bar{\boldsymbol{x}} \colon \bar{U} \to H^n_+(-1)$ defined by

$$\bar{\boldsymbol{x}}(\boldsymbol{x}(u), \varphi) = \cosh\theta \boldsymbol{x}(u) + \sinh\theta(\cos\varphi \boldsymbol{n}(u) + \sin\varphi \boldsymbol{e}(u)),$$

where $\bar{U} = M \times [0, 2\pi)$ and $|\theta|$ a sufficiently small positive real number.

We can define the *hyperbolic height functions family* on CM by

$$\bar{H} \colon CM \times S^{n-1}_+ \to \mathbb{R}; \ \bar{H}((\boldsymbol{x}(u), \varphi), \boldsymbol{v}') = \langle \bar{\boldsymbol{x}}(\boldsymbol{x}(u), \varphi), \boldsymbol{v}' \rangle.$$

We denote that $\bar{h}_{v_0}(u, \varphi) = \bar{H}((\boldsymbol{x}(u), \varphi), \boldsymbol{v}_0)$, for $\boldsymbol{v}_0 \in S^{n-1}_+$. The following proposition characterizes the singular points of the hyperbolic height functions over CM.

Proposition 3.2. *Let CM be a hyperbolic canal hypersurface of M in $H^n_+(-1)$ and $\bar{H} \colon CM \times S^{n-1}_+ \longrightarrow \mathbb{R}$ the hyperbolic height function on CM. Then $(\partial\bar{h}_v/\partial u_i)(u, \varphi) = (\partial\bar{h}_v/\partial\varphi)(u, \varphi) = 0$ ($i = 1, \ldots, n-2$) if and only if $\boldsymbol{v} = \widetilde{\boldsymbol{x} + \boldsymbol{w}}(u, \varphi)$, where $\boldsymbol{w}(u, \varphi) = \cos\varphi \boldsymbol{n}(u) + \sin\varphi \boldsymbol{e}(u)$ and $0 \leq \varphi < 2\pi$.*

Proof. We have

$$\bar{\boldsymbol{x}}(u, \varphi) = \cosh\theta \boldsymbol{x}(u) + \sinh\theta(\cos\varphi \boldsymbol{n}(u) + \sin\varphi \boldsymbol{e}(u)) \in \langle \boldsymbol{x}(u), \boldsymbol{n}(u), \boldsymbol{e}(u) \rangle_{\mathbb{R}}.$$

Hence,

$$\begin{aligned} \bar{\boldsymbol{x}}_i(u, \varphi) \ &= \ \cosh\theta \boldsymbol{x}_i(u) + \sinh\theta(\cos\varphi \boldsymbol{n}_i(u) + \sin\varphi \boldsymbol{e}_i(u)), \\ \bar{\boldsymbol{x}}_\varphi(u, \varphi) \ &= \ \sinh\theta(-\sin\varphi \boldsymbol{n}(u) + \cos\varphi \boldsymbol{e}(u)). \end{aligned}$$

We denote

$$\boldsymbol{N}(u,\varphi)=\sinh\theta\boldsymbol{x}(u)+\cosh\theta\boldsymbol{w}(u,\varphi)\in S^n_1,$$

where $\boldsymbol{w}(u,\varphi)=\cos\varphi\boldsymbol{n}(u)+\sin\varphi\boldsymbol{e}(u)\in S^1(N_pM\cap T_pH^n_+(-1))$. It follows that $\boldsymbol{w}_i(u,\varphi)=\cos\varphi\boldsymbol{n}_i(u)+\sin\varphi\boldsymbol{e}_i(u)\in\langle\boldsymbol{x}_1(u),\dots,\boldsymbol{x}_{n-2}(u),\boldsymbol{n}(u),\boldsymbol{e}(u)\rangle_{\mathbb{R}}$ and $\boldsymbol{w}_\varphi(u,\varphi)=-\sin\varphi\boldsymbol{n}(u)+\cos\varphi\boldsymbol{e}(u)$, where $\boldsymbol{w}_i(u,\varphi)=(\partial\boldsymbol{w}/\partial u_i)(u,\varphi)$ and $\boldsymbol{w}_\varphi(u,\varphi)=(\partial\boldsymbol{w}/\partial\varphi)(u,\varphi)$. It is easy to calculate that $\langle\boldsymbol{N}(u,\varphi),\bar{\boldsymbol{x}}(u,,\varphi)\rangle=0$. Moreover, we have

$$\begin{aligned}\langle\boldsymbol{N}(u,\varphi),\bar{\boldsymbol{x}}_i(u,\varphi)\rangle &= \langle\sinh\theta\boldsymbol{x}(u)+\cosh\theta\boldsymbol{w}(u,\varphi),\\ &\quad\ \cosh\theta\boldsymbol{x}_i(u,\varphi)\ +\ \sinh\theta\boldsymbol{w}_i(u,\varphi)\rangle\\ &= \langle\cosh\theta\boldsymbol{w}(u,\varphi),\cosh\theta\boldsymbol{x}_i(u,\varphi)\rangle\\ &= \cosh^2\theta\langle(\cos\varphi\boldsymbol{n}(u)+\sin\varphi\boldsymbol{e}(u)),\boldsymbol{x}_i(u,\varphi)\rangle=0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle\boldsymbol{N}(u,\varphi),\bar{\boldsymbol{x}}_\varphi(u,\varphi)\rangle &= \langle\sinh\theta\boldsymbol{x}(u)+\cosh\theta\boldsymbol{w}(u,\varphi),\sinh\theta\boldsymbol{w}_\varphi(u,\varphi)\rangle\\ &= \langle\sinh\theta\boldsymbol{x}(u),\sinh\theta\boldsymbol{w}_\varphi(u,\varphi)\rangle\\ &= \sinh^2\theta\langle\boldsymbol{x}(u),-\sin\varphi\boldsymbol{n}(u)+\cos\varphi\boldsymbol{e}(u)\rangle=0.\end{aligned}$$

This means that $\boldsymbol{N}(u,\varphi)=\sinh\theta\boldsymbol{x}(u)+\cosh\theta\boldsymbol{w}(u,\varphi)\in S^n_1\cap N_{\bar{p}}CM$, where $\bar{\boldsymbol{x}}(u,\varphi)=\bar{p}$. Thus $\{\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}_1,\dots,\bar{\boldsymbol{x}}_{n-2},\bar{\boldsymbol{x}}_\varphi,\boldsymbol{N}\}$ is a basis of the vector space $T_{\bar{p}}\mathbb{R}^{n+1}_1$. Hence there exist real numbers $\lambda,\ \lambda_i,\ \mu\ (i=1,\dots,n-2)$ and ρ, such that

$$\boldsymbol{v}=\left(\lambda\bar{\boldsymbol{x}}+\sum_{i=1}^{n-2}\lambda_i\bar{\boldsymbol{x}}_i+\mu\bar{\boldsymbol{x}}_\varphi+\rho\boldsymbol{N}\right)(u,\varphi).$$

Therefore $(\partial\bar{h}_v/\partial u_i)(u,\varphi)=(\partial\bar{h}_v/\partial\varphi)(u,\varphi)=0\ (i=1,\dots,n-2)$ if and only if $\langle\bar{\boldsymbol{x}}_i(u,\varphi),\boldsymbol{v}\rangle=\langle\bar{\boldsymbol{x}}_\varphi(u,\varphi),\boldsymbol{v}\rangle=0\ (i=1,\dots,n-2)$ and $\boldsymbol{v}\in S^{n-1}_+$. This is equivalent to the condition that $\lambda_i=\mu=0$, $\lambda=\rho$ and

$$\boldsymbol{v}=\lambda(\widetilde{\bar{\boldsymbol{x}}+\boldsymbol{N}})(u,\varphi)=\widetilde{\bar{\boldsymbol{x}}+\boldsymbol{N}}(u,\varphi)=\widetilde{\boldsymbol{x}+\boldsymbol{w}}(u,\varphi)\in S^{n-1}_+\cap N_{\bar{p}}CM.\quad\square$$

We now define the hyperbolic Gauss-Kronecker curvature of hypersurface in $H^n_+(-1)$ (cf., [2]). Let $\boldsymbol{x}:V\longrightarrow H^n_+(-1)$ be an embedding, where $V\subset\mathbb{R}^{n-1}$ is an open subset. We denote that $S=\boldsymbol{x}(V)$ and identify S and V through the embedding $\boldsymbol{x}$. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \equiv -1$, we have $\langle \boldsymbol{x}_i, \boldsymbol{x}\rangle \equiv 0 \quad (i = 1, \dots, n-1)$, where $u = (u_1, \dots u_{n-1}) \in V$. Define a vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \cdots \wedge \boldsymbol{x}_{n-1}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \cdots \wedge \boldsymbol{x}_{n-1}(u)\|},$$

then we have $\langle \boldsymbol{e}, \boldsymbol{x}_i\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{x}\rangle \equiv 0$ and $\langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv 1$. Therefore the vector $\boldsymbol{x} + \boldsymbol{e}$ is lightlike. Since $\boldsymbol{x}(u) \in H^n_+(-1)$ and $\boldsymbol{e}(u) \in S^n_1$, we can show that $\boldsymbol{x}(u) + \boldsymbol{e}(u) \in LC^*_+$. We define the *hyperbolic Gauss indicatrix* (or the *lightcone dual*) of $\boldsymbol{x}$ as the map

$$\mathbb{L}\colon V \longrightarrow LC^*_+$$

given by $\mathbb{L}(u) = \boldsymbol{x}(u) + \boldsymbol{e}(u)$. We also define the *hyperbolic Gauss map* of $\boldsymbol{x}$ as the map

$$\widetilde{\mathbb{L}}\colon V \longrightarrow S^{n-1}_+$$

given by $\widetilde{\mathbb{L}}(u) = \widetilde{\boldsymbol{x} + \boldsymbol{e}}(u)$. In [2] we have shown that $D_v\boldsymbol{e} \in T_pS$ for any $p = \boldsymbol{x}(u_0) \in S$ and $\boldsymbol{v} \in T_pS$. Here, D_v denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. Therefore, we have $D_v\mathbb{L} \in T_pS$. Under the identification of V and S, the derivative $d\boldsymbol{x}(u_0)$ can be identified to the identity mapping id_{T_pS} on the tangent space T_pS, where $p = \boldsymbol{x}(u_0)$. This means that $d\mathbb{L}(u_0) = id_{T_pS} + d\boldsymbol{e}(u_0)$. Thus, $d\mathbb{L}(u_0)$ can be regarded as a linear transformation on the tangent space T_pS. We call the linear transformation $S_p = -d\mathbb{L}(u_0)\colon T_pS \longrightarrow T_pS$ the *hyperbolic shape operator* of $S = \boldsymbol{x}(V)$ at $p = \boldsymbol{x}(u_0)$. An eigenvalue of S_p is called a *principal hyperbolic curvature* of $\boldsymbol{x}(V) = S$ at $p = \boldsymbol{x}(u_0)$ and denoted by $\bar{\kappa}_p$. The *hyperbolic Gauss-Kronecker curvature* of $S = \boldsymbol{x}(V)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K_h(u_0) = \det S_p.$$

We also denote the hyperbolic Gauss map of CM as the map,

$$\widetilde{\mathbb{L}}\colon CM \longrightarrow S^{n-1}_+,\ \widetilde{\mathbb{L}}(u, \varphi) = \widetilde{\boldsymbol{x} + \boldsymbol{w}}(u, \varphi).$$

Let $\pi : M \times S^{n-1}_+ \longrightarrow S^{n-1}_+; (\boldsymbol{x}(u), \boldsymbol{v}) \mapsto \boldsymbol{v}$ be the natural projection. Clearly, $\pi|CM = \widetilde{\mathbb{L}}$ and the Boardman symbols of DH and $\widetilde{\mathbb{L}}$ satisfy the relation

$$\sum\nolimits^{n-2, r_1, r_2, \dots, r_k}(DH) = \sum\nolimits^{r_1, r_2, \dots, r_k}(\widetilde{\mathbb{L}}).$$

Since CM is a hypersurface in $H^n_+(-1)$, the Proposition 5.2 in [2] includes the following result:

Proposition 3.3. *The set of singular points of the hyperbolic Gauss map $\widetilde{\mathcal{L}}$ of the canal hypersurface CM coincides with the set of horospherical parabolic points (that is, $K_h(\bar{p}) = 0$ at $\bar{p} \in CM$).*

On the other hand, given $(\boldsymbol{x}(u), \boldsymbol{v}) \in CM$, we have $\bar{h}_{v_0}(u, \boldsymbol{v}) = \langle \cosh\theta \boldsymbol{x}(u) + \sinh\theta \boldsymbol{w}, \boldsymbol{v}_0\rangle$, and it is possible to choose a coordinate system $U \times W$ for CM at $(\boldsymbol{x}(u), \boldsymbol{v})$ in such a way that:

a) U is an orthogonal coordinate system for M at $\boldsymbol{x}(0) = (1, 0, \dots, 0)$, and

b) W is a coordinate system for S^{n-1}_+ at $\boldsymbol{v} = (1, 0, \dots, 0, 1)$. And

(1) $\boldsymbol{x}(u) = (f_1(u), u_1, u_2, \dots, u_{n-2}, f_2(u), f_3(u))$;

(2) $(\partial f_i/\partial u_j)(0) = (f_i)_j(0) = 0$, for $i = 2, 3$; $j = 1, 2, \dots, n-2$.

In this case, $(\partial f_1/\partial u_j)(0) = (f_1)_j(0) = 0$ for $j = 1, \dots, n-2$, by $\boldsymbol{x}(u) \in H^n_+(-1)$.

All the events are actually at a neighbourhood of a arbitrary fixed point of CM which serves as the origin of the system. Then we have the following proposition:

Proposition 3.4. *Under the above conditions, we have that the Hessian matrices of hyperbolic height functions in the normal direction $\boldsymbol{v}$ on M (h_v) at $\boldsymbol{x}(u)$ and on CM $(\bar{h}_v)$ at $(\boldsymbol{x}(u), \boldsymbol{v})$ respectively, satisfy $\lambda\mathcal{H}(h_v)(u) \oplus \mu I_1 = \mathcal{H}(\bar{h}_v)(u, \boldsymbol{v}) = D\widetilde{\mathcal{L}}(u, \boldsymbol{v})$, where I_1 denotes the identity over $\mathbb{R}$, λ, μ are nonzero scalars and $D\widetilde{\mathcal{L}}(u, \boldsymbol{v})$ represents the Jacobian matrix of the hyperbolic Gauss map $\widetilde{\mathcal{L}}$ at the point $(\boldsymbol{x}(u), \boldsymbol{v})$.*

Proof. For $(\boldsymbol{x}(u), \boldsymbol{v}) \in CM$

$$h_v(u) = -f_1(u) + \sum_{i=1}^{n-2} v_i u_i + v_{n-1} f_2(u) + v_n f_3(u),$$

where $u = (u_1, \dots, u_{n-2})$ and $\boldsymbol{v} = (1, v_1, \dots, v_n)$. Since $\boldsymbol{v}$ is a pseudo normal vector of M (In fact, for fixed $\boldsymbol{n}$, $\boldsymbol{v}$ is a normal vector of CM by Propositions 3.1 and 3.2),

$$v_i = \frac{\partial f_1}{\partial u_i}(u) - \frac{\partial f_2}{\partial u_i}(u) v_{n-1}(u) - \frac{\partial f_3}{\partial u_i}(u) v_n(u) \ (i = 1, \dots, n-2).$$

Since $\boldsymbol{v} \in S_+^{n-1}$, we have $\sum_{i=1}^{n} v_i^2 = 1$, so that $v_i = v_i(u, v_{n-1})$ $(i = 1, \dots, n-2)$ and

$$v_n = \frac{-(cv_{n-1} - d) \pm \sqrt{(cv_{n-1} - d)^2 - a(bv_{n-1}^2 - 2ev_{n-1} + f)}}{a},$$

where

$$a = \sum_{i=1}^{n-2} ((f_3)_i(u))^2 + 1, \qquad b = \sum_{i=1}^{n-2} ((f_2)_i(u))^2 + 1,$$

$$c = \sum_{i=1}^{n-2} (f_2)_i(u) \cdot (f_3)_i(u), \qquad d = \sum_{i=1}^{n-2} (f_1)_i(u) \cdot (f_3)_i(u),$$

$$e = \sum_{i=1}^{n-2} (f_1)_i(u) \cdot (f_2)_i(u), \qquad f = \sum_{i=1}^{n-2} ((f_1)_i(u))^2 - 1.$$

Now we choose $\boldsymbol{w} = (0, v_1, \dots, v_n)$ such that $\boldsymbol{v} = \boldsymbol{x}\widetilde{(u) +} \boldsymbol{w}$. Then

$$\begin{aligned} \bar{h}_{\bar{v}}(u, \boldsymbol{v}) &= \langle \cosh\theta \boldsymbol{x}(u) + \sinh\theta \boldsymbol{w}, \bar{\boldsymbol{v}} \rangle \\ &= \cosh\theta h_{\bar{v}}(u) + \sinh\theta \langle \boldsymbol{w}, \bar{\boldsymbol{v}} \rangle \\ &= \cosh\theta h_{\bar{v}}(u) + \sinh\theta \sum_{i=1}^{n} v_i \bar{v}_i, \end{aligned}$$

where $\bar{\boldsymbol{v}} = (1, \bar{v}_1, \dots, \bar{v}_n)$ in S_+^{n-1}. Therefore, under the conditions that $\bar{\boldsymbol{v}} = \boldsymbol{v} = (1, 0, \dots, 0, 1)$ and $\boldsymbol{x}(0) = (1, 0, \dots, 0)$, the Hessian matrix of $\mathcal{H}(\bar{h}_v)(0, \boldsymbol{v})$ has the following form:

$$\mathcal{H}(\bar{h}_v)(0, \boldsymbol{v}) = \begin{pmatrix} \cosh\theta(-1 + f_3(0))_{11} & \cdots & \cosh\theta(f_3(0))_{1\, n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \cosh\theta(-1 + f_3(0))_{1n-2} & \cdots & \cosh\theta(f_3(0))_{n-2\, n-2} & 0 \\ 0 & \cdots & 0 & \sinh\theta \end{pmatrix}_{u=0}$$

$= \cosh\theta \mathcal{H}(h_v)(0) \oplus \sinh\theta I_1 = \cosh\theta \mathcal{H}(h_v)(0) \oplus \sinh\theta I_1 = \mathcal{H}(\bar{h}_v)(0, \boldsymbol{v}) = D\widetilde{\mathcal{L}}(0, \boldsymbol{v})$. □

It follows that $(\boldsymbol{x}(u), \boldsymbol{v}) \in \sum^r(\widetilde{\mathcal{L}})$ if and only if $\boldsymbol{x}(u)$ is a singularity of corank $h_v(u) = r$. In particular, the condition $(\boldsymbol{x}(u), \boldsymbol{v}) \in \sum^{1_k,0}(\widetilde{\mathcal{L}})$ is equivalent to the condition that $\boldsymbol{x}(u)$ is a singularity of type A_k for h_v. As a corollary of Propositions 3.3 and 3.4, we have the following proposition:

Proposition 3.5. *Let $H: U \times S^{n-1}_{+} \longrightarrow \mathbb{R}$ be a hyperbolic height function of M, then $\boldsymbol{x}(u) \in M$ is a degenerate singularity of h_v if and only if $(\boldsymbol{x}(u), \boldsymbol{v})$ is a singular point of $\widetilde{\mathcal{L}}$ or equivalent to the condition that $K_h(u, \boldsymbol{v}) = 0$, where $K_h(u, \boldsymbol{v})$ is the hyperbolic Gauss-Kronecker curvature of CM.*

Throughout the remainder in this paper, we shall assume that all the singularities are corank 1, that is A_k type unless the contrary is explicitly stated.

For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define the *hyperhorosphere* with lightlike pseudo normal $\boldsymbol{v}$ by

$$HS^{n-1}(\boldsymbol{v}, c) = H^n_+(-1) \cap HP(\boldsymbol{v}, c).$$

If we choose a lightlike vector $\boldsymbol{v}_0 = -(1/c)\boldsymbol{v} \in LC^*_+$, then $HS^{n-1}(\boldsymbol{v}, c) = HS^{n-1}(\boldsymbol{v}_0, -1)$. We call $\boldsymbol{v}_0$ the *polar vector* of $HS^{n-1}(\boldsymbol{v}_0, -1)$. We say that a hyperhorosphere $HS^{n-1}_{\boldsymbol{v}} := HS^{n-1}(\boldsymbol{v}, -1)$ has *higher order contact* with M at $\boldsymbol{x}(u)$ if it is tangent to M at $\boldsymbol{x}(u)$ and $\boldsymbol{x}(u)$ is a degenerate singular point of the hyperbolic height function h_v. In this case, we say that $\boldsymbol{v}$ is a *horobinormal vector* and $HS^{n-1}_{\boldsymbol{v}}$ is a *osculating hyperhorosphere* of M at $\boldsymbol{x}(u)$. Given a regular submanifold M of $H^n_+(-1)$, a hyperhorosphere $HS^{n-1}_{v_0}$, tangent to M at some point $p = \boldsymbol{x}(u_0)$, is said to be a *locally supporting hyperhorosphere* for M at p provided there exist an open neighbourhood V of p in $H^n_+(-1)$ such that $M \cap V \subset (HS^{n-1}_{v_0})_-$, where $(HS^{n-1}_{v_0})_- = \{\boldsymbol{u} \in H^n_+(-1) \mid \langle \boldsymbol{u} - \boldsymbol{x}(u_0), \boldsymbol{v}\rangle \geq 0\}$. The submanifold M is said to be a *locally horoconvex* at a point $p = \boldsymbol{x}(u)$ if there exist some locally supporting hyperhorosphere for M at p. Then we have the following assertions:

Theorem 3.6. *Let M be an $(n-2)$-submanifold of $H^n_+(-1)$.*

(1) *If n is odd, then M admits at least a horobinormal direction and at most $n-2$ at each one of its points.*

(2) *If n is even and M admits a locally supporting hyperhorosphere at some point, then there is a non empty open submanifold in M, all of whose points admit at least one horobinormal direction and at most $n-2$ of them.*

Proof. Let $H: U \times S^{n-1}_+ \longrightarrow \mathbb{R}$; $(u, \boldsymbol{v}) \mapsto \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle = h_v(u)$ be a hyperbolic height function of M, where $\boldsymbol{v} = (1, v_1, v_2 \dots, v_n) \in S^{n-1}_+$. Then we can locally write

$$h_v(u) = -f_1(u) + \sum_{i=1}^{n-2} v_i u_i + v_{n-1} f_2(u) + v_n f_3(u).$$

Consequently, we have that $\boldsymbol{x}(0) = p \in M$ is a singular point of h_v if and only if $\boldsymbol{v} = (1, 0, \ldots, 0, v_{n-1}, v_n)$.

Denote the 2-jets of the components, $f_i,\ i = 1, 2, 3$, of the embedding $\boldsymbol{x}$ by

$$j^2 f_i(0) = \sum_{k=1}^{n-2} a^i_{kk} u_k^2 + 2\sum_{k<l} a^i_{kl} u_k u_l.$$

Then the Hessian of h_v at the point p is given by $\mathcal{H}(h_v(0)) = 2^{n-2}\ H_p^{n-2}(v_{n-1}, v_n)$, where H_p^{n-2} denotes a polynomial of degree $n-2$ in the two variables v_{n-1}, v_n, whose coefficients depend on the coefficients a^i_{kl} of the 2-jets $j^2 f_i(0),\ i = 1, 2, 3$.

It then follows that p is a degenerate singularity of h_v if and only if v_{n-1}, v_n satisfy the equation $H_p^{n-2}(v_{n-1}, v_n) = 0$. So, the horobinormal directions of M at the point p can be characterized as the roots of polynomial H_p^{n-2}.

Taking into account that $\boldsymbol{v} = (1, 0, \ldots, 0, v_{n-1}, v_n) \in S^{n-1}_+$ and thus $v_{n-1}^2 + v_n^2 = 1$, we can consider $H_p^{n-2}(v_{n-1}, v_n)$ as a polynomial $H_p^{n-2}(a)$ in one variable a. Consider the decomposition of this polynomial over the field $\mathbb{C}$,

$$H_p^{n-2}(a) = (\alpha_1 a - \beta_1)(\alpha_2 a - \beta_2)\ldots(\alpha_{n-2} a - \beta_{n-2}),$$

where $\alpha_i, \beta_i \in \mathbb{C};\ i = 1, 2, \ldots, n-2$. Analising the possible real solutions of $H_p^n(a, b) = 0$ for each $p \in M$ leads to a subdivision of M into regions $M(i_1, \ldots, i_r) = \{p \in M\colon H_p^n$ admits r distinct real roots with respective multiplicities $i_1, \ldots, i_r\}$. We denote by M_k the subset of points of M whose corresponding polynomial has only simple roots, exactly k of them being real, $k = 1, \ldots, n-2$, and by M^k those for which H_p^n has k real roots counted with their corresponding multiplicities, that is,

$$M^k = \cup M(i_1, \ldots, i_r),\ i_1 + \ldots + i_r = k.$$

We observe that M_k is open in M and that $M^k \subseteq cl(M_k)$ (where cl(-) denotes the closure operator). If we put $M^* = \bigcup_{k=1}^n (M_k)$, it follows that $M = cl(M^*) \bigcup M_0$.

We now show that if n is even, $M_0 \neq M$ and hence $M^* \neq \emptyset$. We do this by contradiction. Suppose that $M_0 = M$. In this case, we have that all the hyperbolic height functions on M have non degenerate singularities and hence, so do the hyperbolic height functions on CM. Now, the existence of a locally support hyperhorosphere on M implies that there is at least some point $(x(u), \boldsymbol{v}) \in CM$, such that the Hessian matrix of the hyperbolic height function $\bar{h}_v$ at the point $x(u)$

defines an elliptic quadratic form (that is, of zero signature). But this implies that the Hessian matrices of all the height functions on M define elliptic quadratic forms, for otherwise, there would be some degenerate height function on CM. It follows that we can find locally support hyperhorospheres at every point of CM. Denote by $\pi_1 \colon H^n_+(-1) \to \mathbb{R}^n_0$ the pseudo-orthogonal projection of the hyperbolic n-space to the hyperplane, $\mathbb{R}^n_0$, pseudo-perpendicular to the vector $(1, 0, \dots, 0) \in \mathbb{R}^{n+1}_1$. Clearly, π_1 defines a diffeomorphism of $H^n_+(-1)$ onto the euclidean n-space $\mathbb{R}^n_0$ that transforms hyperhorospheres into parabolic quadric hypersurfaces. Since parabolic quadric hypersurfaces are convex, $\pi_1(CM)$ is a hypersurface in $\mathbb{R}^n_0$ that has a locally support hyperplane at every point. But this implies that it is a strictly convex and thus diffeomorphic to a $(n-1)$-sphere (see [10, Chapter 13]). Consequently CM is also diffeomorphic to a $(n-1)$-sphere, which contradicts the fact that it is a canal hypersurface over the submanifold M.

On the other hand, if n is odd, we have that H^{n-2}_p must admit at least one and at most n real roots, for all $p \in M$. Hence $M_0 = \emptyset$ and $M = cl(M^*)$ and there is at least a binormal direction at each point of M. Clearly, the maximum number of binormal directions at each point is $n-2$, for each one of them comes from a real root of $H^{n-2}_p(a) = 0$. □

We say that a point $p = \boldsymbol{x}(u) \in M$ is a *horospherical ridge point* if there is a vector $\boldsymbol{w} \in S^1(N_pM \cap T_pH^n_+(-1))$ for M at p such that p is a singularity of type $A_{k\geq 3}$ of h_v where $\boldsymbol{v} = \widetilde{\boldsymbol{x}+\boldsymbol{w}}(u,\varphi)$. Moreover, we say that a point $p = \boldsymbol{x}(u) \in M$ is a *kth-order horospherical ridge point* if there is some vector $\boldsymbol{w} \in S^1(N_pM \cap T_pH^n_+(-1))$ for M at p such that p is a singularity of type A_k of h_v where $\boldsymbol{v} = \widetilde{\boldsymbol{x}+\boldsymbol{w}}(u,\varphi)$ and $k \geq 3$.

Proposition 3.7. *Let M be an $(n-2)$-submanifold of $H^n_+(-1)$. Then a point $\boldsymbol{x}(u)$ is a horospherical ridge point of M if and only if $(\boldsymbol{x}(u), \boldsymbol{v})$ is a non stable singularity of $\widetilde{\mathcal{L}}(\boldsymbol{x}(u), \boldsymbol{v})$.*

Proof. By Proposition 3.3, the point $\boldsymbol{x}(u)$ is a degenerate singularity of h_v if and only if $(\boldsymbol{x}(u), \boldsymbol{v})$ is a singular point of the Gauss map $\widetilde{\mathcal{L}}(\boldsymbol{x}(u), \boldsymbol{v})$. It follows from the definition of $\widetilde{\mathcal{L}} \colon CM \to S^{n-1}_+$ that $(\boldsymbol{x}(u), \boldsymbol{v}) \in \Sigma^{1,1}(\widetilde{\mathcal{L}})$, (where Σ^i denotes the i-th Boardman symbol, $i > 0$) if and only if $\boldsymbol{x}(u)$ is a singularity of type $A_{k\geq 3}$ of h_v. □

For a generic M, the subset $\Sigma^1(\widetilde{\mathcal{L}})$, being the regular part of the subset ${K_h}^{-1}(0)$, is an $(n-2)$-submanifold of CM (c.f., [5]). Consider the natural projection

$\nu\colon CM \to M$ and its restriction $\bar{\nu} = \nu|_{\Sigma^1(\widetilde{\mathcal{L}})}\colon \Sigma^1(\widetilde{\mathcal{L}}) \to M$. Let

$$\bar{M}^* = \bar{\nu}^{-1}(M^*) = \bigcup_{k=1}^{n-2} \bar{M}_k,$$

where $\bar{M}_k = \bar{\nu}^{-1}(M_k)$. Clearly, $\bar{M}^*$ is an open submanifold of ${K_h}^{-1}(0)$, and thus has dimension $n-2$.

Lemma 3.8. *The map $\bar{\nu}|_{\bar{M}^*}\colon \bar{M}^* \to M^*$ is a local diffeomorphism. Moreover, $\bar{\nu}|_{\bar{M}_k}\colon \bar{M}_k \to M_k$ is a k-fold covering map for each $k = 1, \dots, n-2$.*

Proof. Take coordinates on M and CM as in the proof of Proposition 3.4, so that a degenerate singular point $p \in M$ of a hyperbolic height function h_v has associated to the polynomial $H_p^{n-2}(a)$, where $\boldsymbol{v} = (1, 0, \dots, 0, a, b) \in S_+^{n-1}$ and $a^2 + b^2 = 1$. Observe that $K_h(p, v) = H_p^{n-2}(a) = 0$. Then a is a simple root if and only if $(\partial K_h/\partial a)(p, \boldsymbol{v}) \neq 0$. Since $\bar{M}^*$ is an open submanifold of ${K_h}^{-1}(0)$, the condition $(\partial K_h/\partial a)(p, \boldsymbol{v}) \neq 0$ is equivalent to $\bar{\nu}|_{\bar{M}^*}$ being a diffeomorphism in a neighbourhood of $(p, \boldsymbol{v})$ for any $p \in \bar{M}^*$.

Suppose now that $(p, \boldsymbol{v}) \in \bar{M}_k$, so $p \in M_k$. Then there exist $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in S_+^{n-1}$, such that $(p, \boldsymbol{v}_i) \in \bar{M}_k \subset \bar{M}^*$, with $\boldsymbol{v} = \boldsymbol{v}_j$ for some $j = 1, \dots, k$. Since $\bar{\nu}$ is a diffeomorphism in a neighbourhood of each $(p, \boldsymbol{v}_i)$, we get that $\bar{\nu}|_{\bar{M}_k}\colon \bar{M}_k \to M_k$ is a k-fold covering map over M_k. □

Proposition 3.9. *Let M be a generic regular $(n-2)$-submanifold of $H_+^n(-1)$. Then the horospherical ridge points define a stratified subset $\mathcal{L}$ of M such that $\mathcal{L} \cap M^*$ is an immersed submanifold of codimension one in M.*

Proof. Given a point $p = \boldsymbol{x}(u) \in M$, the condition $(p, v) \in \Sigma^1(\widetilde{\mathcal{L}})$, or equivalently $H_p^{n-2}(v) = 0$, determines locally the point (p, v) as a function of u. Moreover, the condition $(p, v) \in \Sigma^{1,1}(\widetilde{\mathcal{L}})$ defines a relation among the coefficients of the 2-jet of the immersion $\boldsymbol{x}$ at u. This relation determines an algebraic variety V of codimension one in the 2-jet space $J^2(\mathbb{R}^{n-2}, \mathbb{R}^n)$. Clearly, $\mathcal{L} = (j^2\boldsymbol{x})^{-1}(u)$, and then Thom Transversality Theorem ([1]) ensures that, for a generic embedding $\boldsymbol{x}$, $\mathcal{L}$ is a stratified subset of codimension one in M. Moreover, for a generic M, the subset $\Sigma^{1,1}(\widetilde{\mathcal{L}})$ is a submanifold of codimension one in $\Sigma^1(\widetilde{\mathcal{L}})$ and since $\bar{M}^*$ is open in $\Sigma^1(\widetilde{\mathcal{L}})$, we have that $\Sigma^{1,1}(\widetilde{\mathcal{L}}) \cap \bar{M}^*$ is an $(n-3)$-submanifold of $\bar{M}^*$. It then follows from Lemma 3.8 that $\mathcal{L} \cap M^* =$

$\bar{\nu}(\Sigma^{1,1}(\widetilde{\mathcal{L}}) \cap \bar{M}^*)$ is a submanifold of codimension one of M. We observe that, due to the fact that $\bar{\nu}$ is only a local diffeomorphism, the subset $\bar{\nu}(\Sigma^{1,1}(\widetilde{\mathcal{L}}) \cap \bar{M}^*)$ may have (generically transverse) self-intersections. □

4 Horoasymptotic lines and horospherical ridges

Given $p = \boldsymbol{x}(u) \in M$ and a horobinormal vector $\boldsymbol{v}$ at p, we have that p is a degenerate singularity of the height function h_v, and thus, the Hessian matrix $\mathcal{H}(h_v)(u)$ determines a degenerate quadratic form on T_pM. The directions $\alpha \in T_pM$ lying in the kernel of this quadratic form are called *horoasymptotic directions* of M at p.

We observe that if $\boldsymbol{v}$ is a horobinormal vector at p, then we have $K_h(p, \boldsymbol{v}) = 0$, so that $(p, \boldsymbol{v})$ is a *horospherical parabolic point* of CM. In this case, since $(p, \boldsymbol{v})$ is a singular point of $\widetilde{\mathcal{L}}$, there exists a vector $\bar{\alpha} \in \operatorname{Ker} D\widetilde{\mathcal{L}} \subseteq T_{(p,\boldsymbol{v})}CM$. Such a direction is a principal direction on CM whose corresponding principal hyperbolic curvature vanishes at $(p, \boldsymbol{v})$. Clearly, $D\nu(\bar{\alpha})$ is a horoasymptotic direction of M at p, for $D\nu\colon T_{(p,\boldsymbol{v})}CM \to T_pM$ takes the kernel of $D\widetilde{\mathcal{L}}(u, \boldsymbol{v})$ to the kernel of the Hessian of the hyperbolic height function h_v at u.

An immediate consequence of Theorem 3.6 is the following corollary:

Corollary 4.1. *Let M be an $(n-2)$-submanifold of $H^n_+(-1)$.*

(1) *If n is odd, then M admits at least a horoasymptotic direction and at most $n-2$ at each one of its points.*

(2) *If n is even and M admits a locally supporting hyperhorosphere at some point, then there is a non empty open submanifold M^* in M, all of whose points admit at least one horoasymptotic direction and at most $n-2$ of them.*

The horoasymptotic directions define line fields on the open subset M^*. We observe that exactly k asymptotic lines pass through each point of the open submanifold M_k, $k = 1, \dots, n-2$.

For any $p = \boldsymbol{x}(u) \in M$ and any unit vector $\alpha \in T_pM$, we define the *normal section*, γ_α, of M as the intersection

$$\gamma_\alpha = M \cap \langle N_pM, \alpha \rangle_{\mathbb{R}} \subset M \subset H^n_+(-1).$$

Clearly, γ_α can be regarded as a curve in hyperbolic 3-space $H^3_+(-1)$.

The horospherical geometry of curves immersed in Hyperbolic 3-space has been studied in [3]. We quote here some of the concepts introduced in [3] that shall be used in this paper.

Given a curve $\gamma : I \to H^3(-1)$, that we can assume parametrised by arc-length, we can define a pseudo-orthonormal frame $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ for $\mathbb{R}^4_1$ along γ. This satisfies the following Frenet-Serre type formulae:

$$\begin{cases} \gamma'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= k_h(s)\boldsymbol{n}(s) + \gamma(s) \\ \boldsymbol{n}'(s) &= -k_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) &= -\tau_h(s)\boldsymbol{n}(s) \end{cases}$$

where $k_h(s) = ||\boldsymbol{t}'(s) - \gamma(s)||$ and $\tau_h(s) = -\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))/(k_h(s))^2$.

We define the following function on the curve:

$$\sigma_h(s) = \left((k_h')^2 - (k_h)^2(\tau_h)^2\right)((k_h)^2 - 1)(s).$$

It was shown in [3] that $\sigma_h(s) = 0$ if and only if the curve γ has a higher order contact with a horosphere at $\gamma(s)$. Consequently, we call σ_h the *horospherical torsion function* of γ.

Theorem 4.2. *Let M be a regular $(n-2)$-submanifold M of $H^n_+(-1)$. Take a horoasymptotic direction α at a point $\boldsymbol{x}(u) \in M$ and let γ_α be the normal section of M in the direction α at $\boldsymbol{x}(u_0) = \gamma_\alpha(0)$, parametrised by arclength. Suppose that $k_h(0) \neq 0$ then we have that $\boldsymbol{x}(u_0) = \gamma_\alpha(0)$ is a 3rd-order horospherical ridge point of M if and only if $\sigma_h(0) = 0$. Moreover, $\boldsymbol{x}(u_0) = \gamma_\alpha(0)$ is a kth-order horospherical ridge point of M if and only if $\sigma_h(0) = \sigma_h'(0) = \ldots = \sigma_h^{(k-3)}(0) = 0$.*

Proof. Since γ_α is a normal section of M associated to the horoasymptotic direction α and $k_h(0) \neq 0$, we have that $\alpha \in \mathrm{Ker}\,(\mathcal{H}(h_{v_0}(u_0)))$. Moreover, the function $h_{v_0}^{\gamma_\alpha} = h_{v_0}|_{\gamma_\alpha} : H^3_+(-1) \supset \gamma_\alpha \to \mathbb{R}$ is a hyperbolic height function on γ_α, where $\boldsymbol{v}_0$ is a horobinormal vector of M at $\boldsymbol{x}(u_0)$ associated to the direction α. Hence $\boldsymbol{x}(u_0)$ is a $A_{k\geq 3}$ singularity of h_{v_0} if and only if $\boldsymbol{x}(u_0)$ is a $A_{k\geq 3}$ singularity of $h_{v_0}^{\gamma_\alpha}$. But it follows from Proposition 3.1 in [3], that this is equivalent to the vanishing of the horospherical torsion of γ_α at 0 and its derivatives up to order $(k-3)$. □

The *horospherical surface* of a curve $\gamma : I \to H^3(-1)$ was defined in [3] as the image of the map $HS_{\gamma_\alpha} : I \times [0, 2\pi) \longrightarrow LC^*_+ \subset \mathbb{R}^4_1$ given by $(s, \varphi) \mapsto \gamma_\alpha(s) + \boldsymbol{w}(s, \varphi) = \boldsymbol{x}(u(s)) + \boldsymbol{w}(s, \varphi)$, with $\boldsymbol{w}(s, \varphi) = \cos\varphi \boldsymbol{n}(s) + \sin\varphi \boldsymbol{e}(s)$. An immediate consequence of Theorem 4.2 and Theorem 2.1 in [3] is the following:

Corollary 4.3. *Given a horoasymptotic direction α at a point $\boldsymbol{x}(u)$ of a regular $(n-2)$-submanifold M in $H^n_+(-1)$. Let γ_α be the normal section of M in the direction α at a point $\boldsymbol{x}(u) = \gamma_\alpha(0)$, parametrised by arclength and suppose that $k_h(0) \neq 0$. Then we have that $\boldsymbol{x}(u)$ a 3-order horospherical ridge point if and only if the horospherical surface HS_{γ_α} of γ_α is locally diffeomorphic to the swallow tail SW at the point $(\boldsymbol{x}(u(s_0)), \boldsymbol{v}_0)$.*

We shall see in what follows, how to characterize the horospherical ridges in terms of the contacts of the osculating hyperhorosphere of M with its horoasymptotic lines.

Lemma 4.4. *Let M be a regular $(n-2)$-submanifold of $H^n_+(-1)$ and $\boldsymbol{v}$ a horobinormal for M at noninflection point $\boldsymbol{x}(u)$ (i.e., $\boldsymbol{x}(u)$ is a A_k type singular point of h_v). Then the osculating hyperhorosphere HS^{n-1}_v at $\boldsymbol{x}(u) \in M$ has contact of order at least 2 with the horoasymptotic line associated to $\boldsymbol{v}$ and passing through $\boldsymbol{x}(u)$.*

Proof. Let $\beta = \beta(s)$ be a horoasymptotic line of M with $\beta(0) = \boldsymbol{x}(u) \in M$. To prove the lemma we just need to verify that $\langle \beta'(s), \boldsymbol{v}\rangle = \langle \beta''(s), \boldsymbol{v}\rangle = 0$ (i.e., $(\partial h_v/\partial u_i)(u) = \det \mathcal{H}(h_v)(u) = 0$). We can locally write, $\beta(s) = (f_1(s), u_1(s), \ldots, u_{n-2}(s), f_2(s), f_3(s))$, where M is locally given in the Monge form:

$$\boldsymbol{x}(u) = (f_1(u), u_1, \ldots, u_{n-2}, f_2(u), f_3(u));$$

where $u = (u_1, \ldots, u_{n-2})$, $(\partial f_j/\partial u_i)(0) = (f_j)_i(0) = 0$; $i = 1, \ldots, n-2$; $j = 2, 3$ and $f_1(u) = \sqrt{f_2^2(u) + f_3^2(u) + \sum_{i=1}^{n-2} u_i^2 + 1}$.

Then we have,

$$\begin{aligned}\beta'(s) &= \Bigg(\sum_{i=1}^{n-2}(f_2(f_2)_i u_i' + f_3(f_3)_i u_i' + u_i u_i')(f_1)^{-1}, u_1', \ldots, u_{n-2}', \\ &\quad \sum_{i=1}^{n-2}(f_2)_i u_i', \sum_{i=1}^{n-2}(f_3)_i u_i'\Bigg),\end{aligned}$$

$$
\begin{aligned}
\beta''(s) \;=\;& \Bigg(\sum_{i=1}^{n-2}\Bigg(\sum_{j=1}^{n-2}((f_2)_j(f_2)_i u_i' u_j' + f_2(f_2)_{ij} u_j' u_i' + (f_3)_j(f_3)_i u_i' u_j' \\
&+ f_3(f_3)_{ij} u_j' u_i') + f_2(f_2)_i u_i'' + f_3(f_3)_i u_i'' + (u_i')^2 + u_i u_i''\Bigg)(f_1)^{-1} \\
&- \Bigg(\sum_{i=1}^{n-2}(f_2(f_2)_i u_i' + f_3(f_3)_i u_i' + u_i u_i')\Bigg)^2 (f_1)^{-3}, \\
& u_1'', \dots, u_{n-2}'', \sum_{i=1}^{n-2}(f_2)_i u_i'' + \sum_{i,j=1}^{n-2}(f_2)_{ij} u_i' u_j', \sum_{i=1}^{n-2}(f_3)_i u_i'' \\
&+ \sum_{i,j=1}^{n-2}(f_3)_{ij} u_i' u_j'\Bigg).
\end{aligned}
$$

Let $\mathcal{H}(h_v)(0)$ be the Hessian matrix of hyperbolic height function in the normal direction $\boldsymbol{v}$ on M at $\boldsymbol{x}(0)$ and

$$
\bar{\mathcal{H}}(h_v)(0) = \begin{pmatrix} 1 & 0\cdots 0 & 0 & 0 \\ 0 & & 0 & 0 \\ \vdots & \mathcal{H}(h_v)(0) & \vdots & 0 \\ 0 & & 0 & 0 \\ 0 & 0\cdots 0 & 1 & 0 \\ 0 & 0\cdots 0 & 0 & 1 \end{pmatrix}.
$$

Hence $\langle \beta'(0), \boldsymbol{v}\rangle = 0$ by $\beta'(0) \in T_{\boldsymbol{x}(0)}M$ and $\boldsymbol{v}$ is horobinormal vector at $\boldsymbol{x}(0)$ of M. Also

$$
\begin{aligned}
\langle \beta''(0), \boldsymbol{v}\rangle \;=\;& \sum_{i,j=1}^{n-2}(f_2)_{ij}(0) u_i'(0) u_j'(0) v_{n-1} + \sum_{i,j=1}^{n-2}(f_3)_{ij}(0) u_i'(0) u_j'(0) v_n \\
&- \sum_{i=1}^{n-2} u_i'(0) u_i'(0) = \beta'(0)\bar{\mathcal{H}}(h_v)^t \beta'(0) \\
=\;& \bar{\mathcal{H}}(h_v)(\beta'(0), \beta'(0)) = 0,
\end{aligned}
$$

where $\boldsymbol{x}(0)$ is a degenerate point of h_v, that is, $\boldsymbol{v} = (1, 0, \dots, 0, v_{n-1}, v_n)$. □

Lemma 4.5 (Romero-Fuster and Sanabria-Codesal [8], Lemma 3.2). *Let $h\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function with a degenerate singularity at the origin and suppose that $\alpha \in$ Ker $(\mathcal{H}(h)(0))$. Then we have that 0 is a singularity of type A_k of h if and only if the vector α belongs to the kernel of the k-linear form, $D^k h(0)$, given by the k-th differential of h, $k \geq 2$.*

Theorem 4.6. *Under the conditions of Lemma 4.4, $\boldsymbol{x}(u(0)) = \beta(0)$ is a horospherical ridge point if and only if the osculating hyperhorosphere $HS^{n-1}_{\boldsymbol{v}}$ at $\boldsymbol{x}(u(0)) \in M$ has contact order at least 3 with the horoasymptotic line β.*

Proof. It is enough to show that $\langle \beta'(0), \boldsymbol{v}\rangle = \langle \beta''(0), \boldsymbol{v}\rangle = \langle \beta'''(0), \boldsymbol{v}\rangle = 0$. Now in the above coordinates, we have

$$\begin{aligned}\beta'''(0) \quad = \quad & \Bigg(3\sum_{i=1}^{n-2}(u_i'u_i'')(0), u_1'''(0), \ldots, u_{n-2}'''(0), \sum_{i,j,k=1}^{n-2}((f_2)_{ijk}u_i'u_j'u_k')(0) \\ & + 3\sum_{i,j=1}^{n-2}((f_2)_{ij}u_i''u_j')(0), \sum_{i,j,k=1}^{n-2}((f_3)_{ijk}u_i'u_j'u_k')(0) \\ & + 3\sum_{i,j=1}^{n-2}((f_3)_{ij}u_i''u_j')(0)\Bigg),\end{aligned}$$

where $u_i(0) = 0$, $f_j(0) = 0$, $(f_j)_i(0) = 0$, $i = 1, \ldots, n-2$ and $j = 2, 3$. It follows that

$$\langle \beta'''(0), \boldsymbol{v}\rangle = 3D^2 h_v(0)(\beta'(0), \beta''(0)) + D^3 h_v(0)(\beta'(0), \beta'(0), \beta'(0)).$$

By Lemmas 4.4 and 4.5,

$$\beta'(0) \in \text{Ker } D^2 h_v(0) \cap \text{Ker } D^3 h_v(0) \textit{ and } \langle \beta'(0), \boldsymbol{v}\rangle = \langle \beta''(0), \boldsymbol{v}\rangle = 0.$$

Hence $\langle \beta'(0), \boldsymbol{v}\rangle = \langle \beta''(0), \boldsymbol{v}\rangle = \langle \beta'''(0), \boldsymbol{v}\rangle = 0$. Reversing the above argument yields the converse. □

For a generic regular $(n-2)$-submanifold M of $H^n_+(-1)$, the submanifold $\mathcal{L}$ can be decomposed into a union of $(n-k)$-submanifolds, $\mathcal{L} = \bigcup_{k=3}^n \mathcal{L}_k$, where $\mathcal{L}_k$ denotes the subset of horospherical ridges of order k. The highest order horospherical ridges, $\mathcal{L}_n$ are isolated points on M. We see next how to distinguish among the horospherical ridges of different orders.

Theorem 4.7. *Under the condition of Lemma* 4.4, $\boldsymbol{x}(0) = \beta(0) \in M$ *is a horospherical ridge point of order* k *if and only if the osculating hyperhorosphere* $HS^{n-1}_{\boldsymbol{v}}$ *has contact of order at least* k *with the horoasymptotic line* β.

Proof. The proof runs similarly to that of Theorem 4.6 and we omit the details here. □

Finally, we give a characterization for horospherical ridges of maximal order in terms of the horospherical geometry of the horoasymptotic lines of M considered as curves in hyperbolic n-space. The study of the horospherical properties of curves immersed in $H^3_+(-1)$, made in [3], can be naturally extended to the case of curves immersed into higher dimensional hyperbolic spaces as follows:

Given a curve, $\gamma : I \to H^n(-1)$, that we can assume parametrised by arc-length, we can define a pseudo-orthonormal frame $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}_1(s), \dots, \boldsymbol{n}_{n-1}(s)\}$ for $\mathbb{R}^{n+1}_1$ along γ that satisfies the following Frenet-Serre type formulae.

$$\begin{cases} \gamma'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= k_1(s)\boldsymbol{n}_1(s) + \gamma(s) \\ \boldsymbol{n}'_1(s) &= -k_1(s)\boldsymbol{t}(s) + k_2(s)\boldsymbol{n}_2(s) \\ \cdots &= \cdots \\ \boldsymbol{n}'_i(s) &= -k_i(s)\boldsymbol{n}_{i-1}(s) + k_{i+1}(s)\boldsymbol{n}_{i+1}(s) \\ \cdots &= \cdots \\ \boldsymbol{n}'_{n-2}(s) &= -k_{n-2}(s)\boldsymbol{n}_{n-3}(s) + k_{n-1}(s)\boldsymbol{n}_n(s) \\ \boldsymbol{n}'_{n-1}(s) &= -k_{n-1}(s)\boldsymbol{n}_{n-2}(s) \end{cases}$$

where

$$k_1(s) = \|\boldsymbol{t}'(s) - \gamma(s)\|,$$
$$k_i(s) = \|\boldsymbol{n}'_{i-1}(s) + k_{i-1}\boldsymbol{n}_{i-2}(s)\| \ (i = 2, \dots, n-1, \ \boldsymbol{n}_0(s) = \boldsymbol{t}(s)) \quad \text{and}$$
$$k_{n-1}(s) = -\det(\gamma(s), \gamma'(s), \cdots, \gamma^{(n)}(s))/k_1^{n-1}(s)k_2^{n-2}(s)\cdots k_{n-2}^2(s).$$

Consider the hyperbolic height function on γ,

$$\begin{array}{rccl} H : & I \times S^{n-1}_+ & \longrightarrow & \mathbb{R} \\ & (s, \boldsymbol{v}) & \longmapsto & \langle \gamma(s), \boldsymbol{v} \rangle = h_v(s). \end{array}$$

It is a tedious but straightforward task to show that $h'_v(s_0) = \cdots = h^{(n-1)}_v(s_0) = 0$ if and only if $\boldsymbol{v} = \widetilde{\boldsymbol{v}}_0 \in S^{n-1}_+$, where

$$\boldsymbol{v}_0 = \gamma(s_0) + \sum_{j=1}^{n-2} \sigma_j \boldsymbol{n}_j(s_0) \pm \sqrt{1 - \sum_{j=1}^{n-2} \sigma_j \boldsymbol{n}_{n-1}(s_0)},$$

σ_j, $j = 1, \cdots, n$, are real-valued functions that depend on the functions $\{k_j\}_{j=1}^{n-1}$ and their derivatives. Moreover, $h'_v(s_0) = \cdots = h_v^{(n)}(s_0) = 0$ if and only if $\boldsymbol{v}$ is as above and $\sigma_n(s_0) = 0$. Again, the function σ_n gives a measure of how far the curve γ is from being contained in a hyperhorosphere and will be called *horospherical hypertorsion* of γ.

The *horospherical flattenings* of a curve γ immersed in $H^n(-1)$ are the zeroes of the horospherical hypertorsion of γ.

Now, as a corollary of Theorem 4.7 ,we have the following result.

Corollary 4.8. *A point $\boldsymbol{x}(u) \in M^*$ is a horospherical ridge of maximal order (that is, of order $\geq n$) of M if and only it is a horospherical flattening of some horoasymptotic line of M.*

References

[1] M. Golubitsky and V. Guillemin, *Stable Mappings and their Singularities*. GTM **14** (1973), Springer.

[2] S. Izumiya, D-H. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*. Proceedings of the London Mathematical Society 86 (2003), 485–512.

[3] S. Izumiya, D-H. Pei and T. Sano, *Horospherical surfaces of curves in Hyperbolic space*. To appear in Publ. Math. Debrecen.

[4] S.Izumiya, D-H.Pei and M.C. Romero-Fuster, *The horospherical geometry of surfaces in Hyperbolic* 4*-space*. Preprint.

[5] E.J.N. Looijenga, *Structural stability of smooth families of C^∞-functions*. Thesis, Univ. Amsterdam (1974).

[6] D.K.H. Mochida, R.C. Romero-Fuster and M.A. Ruas, *Osculating hyperplanes and asymptotic directions of codimension two submanifolds of Euclidean spaces*. Geometriae Dedicata, **77** (1999), 305–315.

[7] I. Porteous, *The normal singularities of submanifold*. J. Diff. Geom., **5** (1971), 543–564.

[8] R.C. Romero-Fuster and E. Sanabria-Codesal, *On the flat ridges of submanifolds of codimension 2 in $\mathbb{R}^n$*, Proceedings of the Royal Society of Edinburgh, **132A** (2002), 975–984.

[9] R.C. Romero-Fuster and E. Sanabria-Codesal, *Curvature lines, ridges and conformal invariants of hypersurfaces.* Preprint.

[10] J.A. Thorpe, *Elementary Topics in Differential Geometry.* UTM., (1979) Springer-Verlag.

Shyuichi Izumiya
Department of Mathematics
Hokkaido University
Sapporo 060-0810
JAPAN
e-mail:izumiya@math.sci.hokudai.ac.jp

Donghe Pei
Department of Mathematics
Northeast Normal University
Changchun 130024
P.R. CHINA
e-mail: peidh340@nenu.edu.cn, donghepei@mail.jl.cn

María del Carmen Romero Fuster
Departament de Geometría i Topología
Universitat de València
46100 Burjassot (València)
ESPANYA
e-mail: carmen.romero@post.uv.es

Masatomo Takahashi
Department of Mathematics
Hokkaido University
Sapporo 060-0810
JAPAN
e-mail: takahashi@math.sci.hokudai.ac.jp